RATIONAL MAPS: JULIA SETS FROM ACCESSIBLE MANDELBROT SETS ARE NOT HOMEOMORPHIC

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Abstract. In this paper we investigate the Julia sets of singularly perturbed complex rational maps of the form

\[ F_\lambda(z) = z^n + \frac{\lambda}{z^d} \]

We’ll show that, for the case \( n = d = 2 \), two maps drawn from main cardioids of distinct accessible Mandelbrot sets containing a cycle of period \( m \) do not have homeomorphic Julia sets, unless these cardioids are complex conjugates of one another.

1. Introduction

There has been much work in recent years to understand the dynamics of the family of rational maps on the Riemann sphere, \( \mathbb{C} \), of the form

\[ F_\lambda(z) = z^n + \frac{\lambda}{z^d} \]

with \( n \geq 2 \), \( d \geq 1 \), and \( \lambda \) a complex parameter. The Julia set of such a map is often known to be a Sierpiński curve [2]. Cases for which this is true include: when the critical orbit enters the immediate basin of infinity after two or more iterations, the parameter lies in the main cardioid of a buried baby Mandelbrot set, or the parameter lies on a buried point in a Cantor necklace. When a critical point lies in the immediate basin of attraction of a fixed point, the map has a Checkerboard Julia set [1].

In the case of a Misiurewicz-Sierpiński map, where the critical points are

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strictly pre-periodic to a repelling cycle, the Julia set is known to be a generalized Sierpiński gasket [6].

In this paper, we will consider the case where a critical point lies in the immediate basin of attraction of a periodic point. First, we explore the structure of the Julia sets associated to such maps. Then, for fixed $n$ and $d$, we ask whether two maps, $F_\lambda$ and $F_\mu$, each with a critical point lying in the immediate basin of attraction of a point with period $m$ will have conjugate dynamics on their Julia sets.

2. Preliminaries

For simplicity, we consider here only the cases with $n = d$, thus

\begin{equation}
F_\lambda(z) = z^n + \frac{\lambda}{z^n}.
\end{equation}

When $\lambda = 0$ the map is just the degree-$n$ map, $F_\lambda(z) = z^n$, with a fixed critical point at 0. When $\lambda \neq 0$, the degree of the map doubles to $2n$ and 0 becomes a pole. Then there is a super-attracting critical point at infinity and $2n$ finite critical points located at the $2n^{th}$ roots of $\lambda$. There are, however, only two critical values, $v_\lambda = \pm 2\sqrt{\lambda}$. When $n$ is even, $\pm v_\lambda$ both map onto the same point. When $n$ is odd, the orbits of $\pm v_\lambda$ behave symmetrically with respect to $z \mapsto -z$ under iteration. Thus, we say that there is really only one free critical orbit.

The Julia set of $F_\lambda$, denoted by $J(F_\lambda)$, is defined as the closure of the set of repelling periodic points of $F_\lambda$. The complement of the Julia set is called the Fatou set. Components of the Fatou set may be escaping or non-escaping. That is, orbits of points in the Fatou set may eventually escape to infinity or they may be bounded for all time.

There is an open Fatou component containing infinity that maps $n$-to-1 onto itself. We call this region the immediate basin of attraction and denote it $B_\lambda$. $F_\lambda$ is conjugate to $z^n$ on $B_\lambda$. Since 0 maps onto infinity, there is an open Fatou component containing 0 which also maps $n$-to-1 onto $B_\lambda$. When this region is disjoint from $B_\lambda$, we call it the trapdoor and denote it $T_\lambda$.

We may partition the parameter plane into subsets according to the location of the associated critical values with respect to the sets $B_\lambda$ and $T_\lambda$. The following theorem, from [7], describes all possible cases for which the critical orbit escapes to infinity.

**Theorem 2.1.** (The Escape Trichotomy) Let $v_\lambda$ be a critical value of $F_\lambda$.

1. If $v_\lambda \in B_\lambda$, then $J(F_\lambda)$ is a Cantor set of points.
2. If $T_\lambda \neq B_\lambda$ and $v_\lambda \in T_\lambda$, then $J(F_\lambda)$ is a Cantor set of concentric simple closed curves surrounding the origin. (This does not occur when $n = 2$.)
Otherwise, $J(F_\lambda)$ is a connected set. In particular, if $T_\lambda \neq B_\lambda$ and $F_1^i(v_\lambda) \in T_\lambda$ for some $i \geq 1$, then $J(F_\lambda)$ is a Sierpiński curve.

To illustrate this, we display the parameter plane for $n = d = 4$ in Figure 1. The exterior grey region containing infinity is called the Cantor set locus and corresponds to the first part of the theorem. The light grey region at the center containing $\lambda = 0$ is called the McMullen domain and corresponds to the second part of the theorem. The additional light grey regions are called Sierpiński holes and correspond to the Sierpiński curve Julia sets described in the third part of the theorem.

Also observed in numerically generated diagrams of the parameter plane are small copies of the well-known Mandelbrot set. In many cases, standard arguments using polynomial-like maps (such as in [4]) prove the existence of such baby Mandelbrot sets. When $\lambda$ is chosen from one of these baby Mandelbrot sets, the critical orbit does not escape to infinity and the third part of Theorem 2.1 tells us that the Julia set is connected. Some of these baby Mandelbrot sets are buried, that is, they are not accessible from the Cantor set locus. Julia sets of maps taken from the main cardioids of one of these buried Mandelbrot sets are known to be Sierpiński curves [5]. In this paper, we investigate the structure of Julia sets corresponding to parameters from the accessible baby Mandelbrot sets.
We observe that in each parameter plane with \( n = d > 2 \), there are \( n-1 \) principal accessible Mandelbrot sets [3]. For example, when \( n = d = 4 \), as in Figure 1, we see three large copies of the Mandelbrot set. Maps taken from the main cardioids of these principal Mandelbrot sets are known to have Checkerboard Julia sets [1]. That is, the boundary of each non-escaping Fatou component (colored in black) touches infinitely many boundaries of escaping Fatou components (colored light grey), but it does not touch the boundary of any other non-escaping Fatou component. Similarly, the boundary of any escaping Fatou component touches infinitely many boundaries of non-escaping Fatou components, but it does not touch the boundary of any other escaping Fatou component, as shown in Figure 2. When \( n = d = 2 \), we do not have exact copies of the Mandelbrot set, however, there is a large black cardioid-shaped region. Maps taken from this main cardioid also have Checkerboard Julia sets.

For \( n = d \geq 2 \), any two maps taken from different main cardioids of principal Mandelbrot sets have homeomorphic Julia sets. However, these maps will not necessarily exhibit conjugate dynamics. These facts, as well as a method of determining the exact number of conjugacy classes for such maps, are presented in [1].

If one zooms in on the boundary of the Cantor set locus, one notices additional smaller Mandelbrot sets located symmetrically between the principal Mandelbrot sets. In fact, it appears that there are infinitely many levels of progressively smaller accessible Mandelbrot sets. As in the Checkerboard case, we ask whether maps corresponding to distinct

**Figure 2.** A Checkerboard Julia set corresponding to \( \lambda = 0.15 \) with \( n = d = 4 \) and a magnification.

### 3. Checkerboard Julia Sets
main cardioids from the same level of accessible Mandelbrot sets will have conjugate dynamics. In the case of $n = d = 2$, we will show that the answer is no, unless the accessible Mandelbrot sets are complex conjugates of one another.

4. External Rays in the Parameter Plane

Consider the case where $n = d = 2$. Just as in the case of the complex quadratic map, there exists a homeomorphism, known as the Böttcher coordinate, which takes straight rays extending out from the closed unit disk to rays in the exterior region of the parameter plane [9]. We associate to each external ray an angle between 0 and 1. In particular, the rays having angles of the form $\frac{t}{2m-1}$ where $m = 2, 3, 4, \ldots$ and $t = 1, 2, 3, \ldots, 2m - 2$, land on hyperbolic components containing parameters corresponding to maps with an orbit of period $m$ [10]. We conjecture that these hyperbolic components are, in fact, main cardioids of baby Mandelbrot sets and the external rays land at the cusps of these cardioids. Maps taken from the same hyperbolic component are known to be quasiconformally conjugate on their Julia sets [8], thus any two parameters in the same main cardioid will have attracting periodic orbits of the same period. The unique $\lambda$ in each main cardioid for which the critical orbit is super-attracting is called the center of the main cardioid. Thus the critical orbit of the map taken from the center of the $\frac{t}{2m-1}$ Mandelbrot set has period $m$. We shall restrict our investigation to these points. As an example, the parameter plane and several external rays for $n = d = 2$ are shown in Figure 3.

5. Structure of the Julia Set for $n = d = 2$

The map $F_\lambda(z) = z^2 + \frac{\lambda}{z^2}$ has degree 4 and has four finite critical points. When $\lambda$ is the center of the main cardioid of an accessible Mandelbrot set, one of the four critical points is periodic. The remaining three critical points are pre-periodic and eventually land on the periodic critical orbit. Thus, all four critical points are located in non-escaping Fatou components. Denote the critical point that lies in the first quadrant by $c_1$ and proceed counterclockwise about the origin labeling the others $c_2$, $c_3$, and $c_0$. Then let $C_i$ denote the non-escaping Fatou component containing $c_i$.

Since $F_\lambda^2(z) = F_\lambda^2(iz)$, the Julia set has four-fold symmetry. We conclude that the critical points and corresponding $C_i$ are located symmetrically at quarter-turn intervals about the origin.

By the Escape Trichotomy, we know that the Julia set is connected. Combined with fourfold symmetry, we conclude that $B_\lambda$ is disjoint from $T_\lambda$. Furthermore, since all four critical points lie in non-escaping Fatou
components, $\partial B_\lambda \cap \partial T_\lambda$ cannot contain a critical point, so $\partial B_\lambda$ must be disjoint from $\partial T_\lambda$.

When $\lambda$ is the cusp of the main cardioid of the $\frac{1}{d}$ Mandelbrot set, $F^n_\lambda$ has a unique neutral fixed point located at the intersection of the boundary of a non-escaping Fatou component and $\partial B_\lambda$. Then, for the map taken from the center of the same main cardioid, the boundary of the $C_i$ containing the periodic critical point $c_i$ must intersect $\partial B_\lambda$ at a point $p_i$. Since $\partial T_\lambda$ maps onto $\partial B_\lambda$, the image of $p_i$ under $F_\lambda$ has a pre-image $q_j$ which must be contained in $\partial T_\lambda \cap \partial C_j$, for some $j$. Therefore, each $\partial C_i$ has a point, $p_i$, on $\partial B_\lambda$ and another point, $q_i$, on $\partial T_\lambda$. For this reason, we call $C_i$ a connecting Fatou component (see Figure 4).

$F_\lambda$ maps $B_\lambda$ two-to-one onto itself and also maps $T_\lambda$ two-to-one onto $B_\lambda$. Since $F_\lambda$ has degree 4, this accounts for all possible preimages of points in $B_\lambda$. Thus, any eventually escaping points not already located in $B_\lambda$ must map through $T_\lambda$ before escaping. (It is for this reason we call $T_\lambda$ the trapdoor.) Since the critical orbit is assumed to be periodic, we know that the critical values cannot lie in any pre-image of $T_\lambda$, so these pre-images must map univalently onto $T_\lambda$. We conclude that there are four disjoint preimages of $T_\lambda$ located symmetrically about the origin. We now see that the dynamical plane can be divided into 4 symmetric sectors. Let the sector $I_i$ be defined as the closed region whose boundary is given by the portions of $\partial C_i$, $\partial B_\lambda$, $\partial C_{i+1}$, and $\partial T_\lambda$ that are traced out.
as one moves counterclockwise from $p_i$ to $p_{i+1}$ to $q_{i+1}$ to $q_i$ and back to $p_i$ in such a way that $C_i$ is included in the interior of $I_i$.

As proven in [1], $F_\lambda$ maps each $I_i$ univalently (except at the junction points) over the region that is the complement of the three sets $B_\lambda$, $F_\lambda(C_i)$, and $F_\lambda(C_{i+1})$. More specifically, the portion of $\partial B_\lambda$ in $I_i$ is mapped to half of $\partial B_\lambda$, while the portion of $\partial T_\lambda$ in $I_i$ is mapped to the remaining half of $\partial B_\lambda$. By this result it suffices to focus on only one $I_i$ to understand the entire $J(F_\lambda)$.

Since $\partial T_\lambda$ intersects the boundaries of four connecting Fatou components, the boundary of each preimage of $T_\lambda$ must intersect the boundary of a preimage of each of these four connecting Fatou components. We call these Fatou components the corners of the preimage of $T_\lambda$. We've assumed one $C_i$ with period $m \geq 2$ and all others eventually with period $m$, so no $C_i$ can be its own preimage. Also $F_\lambda(C_1) = F_\lambda(C_3) = -F_\lambda(C_0) = -F_\lambda(C_2)$. Therefore, if $C_1$ maps onto $C_2$, then $C_0$ maps onto itself, a contradiction. By similar arguments, we see that no $C_i$ can be the preimage of any other because this would imply that at least one is fixed. Therefore, all preimages of the $C_i$ are disjoint from the $C_i$. Furthermore, the order of the preimages of the $C_i$, as one travels counterclockwise around the boundary of the preimage of $T_\lambda$, must be the same as the order of the $C_i$ themselves. Thus, each sector can be divided into four subsectors. For $k \geq 1$, we denote a $k^{th}$ preimage of $T_\lambda$ by $T_{s_0...s_{k-1}}^{-k}$, where $s_j = 0, 1, 2, \text{ or } 3$ identifies the sector in which the $j^{th}$ image of $T_{s_0...s_{k-1}}^{-k}$ lands.

We show an example of a Julia set for a map taken from the main cardioid of the $\frac{1}{3}$-Mandelbrot set in Figure 4 and a model of the dynamical plane for the same map in Figure 5.

The following Lemmas are useful in describing the dynamical plane.

**Lemma 5.1.** There cannot be more than four connecting Fatou components.

*Proof.* Without loss of generality, suppose a fifth connecting component, $C_4$, exists within the preimage of sector $I_1$ contained in sector $I_0$. Note that $C_4$ does not contain a critical point. As mentioned earlier, $\partial I_0 \cap \partial B_\lambda$ maps to half of $\partial B_\lambda$ and $\partial I_0 \cap \partial T_\lambda$ maps to the other half of $\partial B_\lambda$. Thus the points $p_4$ and $q_4$, must map to distinct points in $\partial I_1 \cap \partial B_\lambda$ which lie on opposite sides of the point $F_\lambda(p_1) = F_\lambda(q_1)$. Since $\partial B_\lambda$ is invariant under $F_\lambda$, all forward images of $p_4$ and $q_4$ lie in $\partial B_\lambda$. All black Fatou components are preimages of the periodic $C_i$ (where $i = 0, 1, 2, \text{ or } 3$), so there is some minimal $k$ for which $F^k \lambda(C_4) = C_i$. $C_i$ intersects $\partial B_\lambda$ at a single point, $p_i$. Thus $F^k \lambda(p_4) = F^k \lambda(q_4) = p_i$. But $C_4$ and its first $k - 1$ images cannot contain a critical point, so $F^{k-1} \lambda(C_4)$ must map univalently onto $C_i$. Therefore, $F^k \lambda(p_4) \neq F^k \lambda(q_4)$, a contradiction. \qed
Figure 4. The Julia set for $\lambda$ taken from the $\frac{1}{3}$-Mandelbrot set when $n = d = 2$ and a magnification of sector $I_1$.

Figure 5. A model of the dynamical plane for $\lambda$ taken from the $\frac{1}{3}$-Mandelbrot set when $n = d = 2$.

Lemma 5.2. Let $\lambda$ be the center of the main cardioid of an accessible Mandelbrot set with critical orbit of period $m \geq 2$. Then $T_{s_0s_1\ldots s_{k-1}}^{-k}$ (with $1 \leq k \leq m$ and $s_j = 0, 1, 2,$ or $3$) cannot have all four corners lying on $\partial B\lambda$, nor can $T_{s_0s_1\ldots s_{k-1}}^{-k}$ have exactly three corners lying on $\partial B\lambda$.

Proof. By the assumptions $m \geq 2$ and $1 \leq k \leq m$, the corners of $T_{s_0s_1\ldots s_{k-1}}^{-k}$ must be different from $C_{s_0}$ and $C_{s_0+1}$. Suppose there exists $T_{s_0}^{-1}$ with all four corners on $\partial B\lambda \cap \partial I_{s_0}$. Since $\partial B\lambda \cap \partial I_{s_0}$ maps to half
of $\partial B_\lambda$, the images of the four corners of $T_{s_0}^{-1}$, namely the $C_i$, must all lie on the same half of $\partial B_\lambda$. This violates the fourfold symmetry discussed earlier. Now suppose there exists a $T^{-2}_{s_0,s_1}$ having four corners on $\partial B_\lambda$, then its image, namely $T_{s_1}^{-1}$, must also have four corners on $\partial B_\lambda$, but we have shown that this is not possible. Continue inductively. A similar argument will show that $T_{s_0,s_1,...,s_{k-1}}^{{-k}}$ cannot have exactly three corners on $\partial B_\lambda \cap \partial I_{s_0}$. □

Lemma 5.3. For $\lambda$ at the center of the main cardioid of an accessible Mandelbrot set with critical orbit of period greater than or equal to 2, there is exactly one region in each $I_i$ with two corners lying on $\partial B_\lambda$ and two corners lying on $\partial T_\lambda$. Such a region must be $T_{i}^{-1}$.

Proof. Suppose there is a region in $I_i$ with the same structure as $T_{i}^{-1}$. Then its image under $F_\lambda$ will have four corners on $\partial B_\lambda$ but will be different from $T_\lambda$. However, by Lemma 5.2, there are no such regions. □

6. Constructing a Model of the Julia Set for $n = d = 2$

Note that $\overline{F_\lambda}(z) = F_\lambda(\bar{z})$. Therefore, $F_\lambda$ is dynamically conjugate to $F_\lambda$ under complex conjugation and the Julia sets of these maps are homeomorphic. This allows us to restrict our study to the upper-half of the parameter plane and we consider only $\lambda$ at the centers of the main cardioids of the $\frac{1}{2^{m-1}}$ Mandelbrot sets with $t = 1,...,2^{m-1} - 1$.

If the maps $F_\lambda$ and $F_\mu$ have conjugate dynamics on their Julia sets, we should be able to define a homeomorphism between $J(F_\lambda)$ and $J(F_\mu)$. Furthermore, we expect to find a homeomorphism that takes $\partial B_\lambda$ to $\partial B_\mu$ and $\partial T_\lambda$ to $\partial T_\mu$. In the argument below, we construct a basic model for the structure of the Julia set and show that no such homeomorphism can exist between two such models, unless the main cardioids with centers $\lambda$ and $\mu$ are complex conjugates of one another.

We remark that the case where all critical points of $F_\lambda$ are strictly pre-periodic and lie in $\partial B_\lambda$ was studied in [6]. Such maps are known as Misiurewicz-Sierpiński maps or, more succinctly, MS maps. The structure of the dynamical plane corresponding to an MS map is similar to the structure for maps with a periodic critical point. For this reason, we use much of the same notation and a similar method for constructing models of the associated Julia sets.

As in our first example, we begin with $\lambda$ at the center of the main cardioid of the $\frac{1}{4}$-Mandelbrot set. Since $\frac{1}{4}$ has period 2 under the angle-doubling map, we know that one of the critical points of $F_\lambda$ is periodic with period 2. We identify the connecting Fatou components containing the critical points down to single points which we continue to refer to
as $C_i$. Likewise, we identify any images or preimages of the $C_i$ down to single points.

It is known that there exists a unique analytic homeomorphism $h_\lambda$ which takes $B_\lambda$ to the exterior of the closed unit disk, takes $\infty$ to $\infty$, and conjugates $F_\lambda$ in $B_\lambda$ to $z \mapsto z^2$ in $\mathbb{C} - \overline{B}$. Define straight rays in $\mathbb{C} - \overline{B}$ by $R_\theta = \{te^{i\theta} : t > 1\}$ for $0 \leq \theta < 1$. Then the preimages of these rays under $h_\lambda$ define dynamical rays in $B_\lambda$. We then define $H = h_\lambda(F_\lambda^2(c_\lambda))$ where $c_\lambda$ is any of the four critical points. $H$ is an analytic homeomorphism which takes the exterior of the connectedness locus in the parameter plane to $\mathbb{C} - \overline{D}$. Then $H^{-1}(R_\theta)$ defines an external parameter ray. Furthermore, since $F_\lambda$ is hyperbolic on $J(F_\lambda)$, it is known that $\partial B_\lambda$ is a simple closed curve and we can extend this homeomorphism to the boundary ($h_\lambda : B_\lambda \mapsto \mathbb{C} - \overline{D}$).

Suppose $\theta$ is a rational angle between 0 and 1 with period $m$ under the doubling map. Then the parameter ray $H^{-1}(R_\theta)$ lands at a parameter in the boundary of the connectedness locus for which the second iterates of the critical points lie in the immediate basin of a parabolic periodic point on $\partial B_\lambda$ that is the landing point of the dynamical ray $R_\theta$.

Recall that when $n$ is even, all four critical points map onto the same point after two iterations. In particular, $F_\lambda^2(c_\lambda) = 4\lambda + \frac{1}{4}$. Thus, we have a single free critical orbit whose period under $F_\lambda$ is the same as the period under the doubling map of the angle of the parameter ray which lands on the Mandelbrot set containing $\lambda$.

When $\theta = \frac{t}{2^m - 1}$, there are four possible angles for the dynamical rays that land on connecting Fatou components. The one in the first quadrant is $h_\lambda(C_1)$, given by $\frac{t}{2^m - 1}$. The remaining $h_\lambda(C_i)$ (for $i = 2, 3, 0$) are given by the fourfold symmetry.

In our example, $\lambda$ is taken from the $\frac{1}{3}$-Mandelbrot set. We know that the angles associated to the critical orbit are $\frac{1}{4} \mapsto \frac{2}{4} \mapsto \frac{1}{4}$. Since $\frac{1}{4}$ corresponds to the second iterate of the periodic critical point associated to the cycle, we know that the Fatou component containing the periodic critical point must lie at the landing point of the $\frac{1}{4}$-dynamical ray. This implies that $C_2$ contains the periodic critical point, so $h(C_2) = \frac{1}{3}$. By symmetry, we compute that $h(C_1) = \frac{1}{12}$, $h(C_3) = \frac{7}{12}$, and $h(C_0) = \frac{5}{6}$.

Now, the portion of $\partial B_\lambda$ that lies on the boundary of each $I_i$ is associated to an interval on the unit circle. To avoid ambiguity, we take each $C_i$ to lie in $I_i$ but not in $I_{i-1}$.

Thus:

$$h_\lambda(\partial B_\lambda \cap I_1) = \left[ \frac{1}{12}, \frac{1}{3} \right]$$
\[ h_\lambda(\partial B_\lambda \cap I_2) = \begin{bmatrix} 1 & 7 \\ 3 & 12 \end{bmatrix} \]

\[ h_\lambda(\partial B_\lambda \cap I_3) = \begin{bmatrix} 7 & 5 \\ 12 & 6 \end{bmatrix} \]

\[ h_\lambda(\partial B_\lambda \cap I_0) = \begin{bmatrix} 5 & 1 \\ 6 & 12 \end{bmatrix} \].

\(\partial B_\lambda\) is invariant under \(F_\lambda\), so all points in the forward orbit of \(C_2\) lie in \(\partial B_\lambda\). We can now assign an itinerary to \(C_2\) according to the behavior of \(h_\lambda(C_2)\) with respect to the aforementioned intervals on the unit circle. Recall that the forward orbit of the dynamical ray landing at \(C_2\) under doubling was given by \(\frac{1}{3} \mapsto \frac{2}{3} \mapsto \frac{1}{3}\). This tells us that \(C_2 \in \partial I_2, F_\lambda(C_2) \in \partial I_3,\) and \(F_\lambda^2(C_2) = C_2 \in \partial I_2\). We define the itinerary of \(C_2\) by \(S(C_2) = 23\).

We extend the homeomorphism, \(h_\lambda\), to take \(\partial T_\lambda\) to a simple closed curve contained in \(S\), which intersects the unit circle exactly four times at \(\frac{1}{2}, \frac{1}{3}, \frac{7}{12}, \) and \(\frac{5}{6}\). This divides our model into four regions corresponding to the sectors, \(I_i\).

By Lemma 5.3, there is a preimage of \(T_\lambda\) in each \(I_i\), which has two corners on \(\partial B_\lambda\) and two corners on \(\partial T_\lambda\). Using the angle-doubling map, we can compute precisely the angles associated to the corners of each \(h_\lambda(T_i^{-1})\). We remark that \(h_\lambda(F_\lambda(C_2)) = \frac{2}{3}\) corresponds to one of the corners where \(\partial T_3^{-1}\) intersects \(\partial B_\lambda\).

The \(h_\lambda(T_i^{-1})\) divide each \(h_\lambda(I_i)\) into four subsectors which are preimages of the original four sectors. For simplicity and because it causes no confusion, we denote \(h_\lambda(T_i^{-1})\) by \(T_i^{-1}\) and \(h_\lambda(I_i)\) by \(I_i\). We denote any of the preimages of \(I_j\) contained within \(I_i\) by \(I_{ij}^{-1}\). Figure 6 depicts our model as described thus far.

Now each subsector must contain a \(T_{s_0s_1}^{-2}\), which has exactly two corners lying on \(\partial T_{s_0}^{-1}\). We refer to the other two corners of \(T_{s_0s_1}^{-2}\) as the free corners. Since \(F_\lambda^2(C_2) = C_2\), one of the free corners of \(T_{23}^{-2}\) must be \(C_2\). The other free corner of \(T_{23}^{-2}\) must lie in \(\partial I_{23}^{-1} \cap (\partial B_\lambda \cup \partial T_\lambda)\).

To determine the specific location of this corner, we recall that \(F_\lambda\) preserves orientation of points along any preimage of \(\partial B_\lambda\). This allows us to compare the order of the free corners of \(T_{23}^{-2}\) along \(\partial I_{23}^{-1} \cap (\partial B_\lambda \cup \partial T_\lambda)\) to the order of their forward images along \(\partial I_3\). The point where \(\partial T_2^{-1}\) intersects \(\partial I_{23}^{-1} \cap \partial B_\lambda\) maps to \(C_0 = \frac{5}{6}\), the point where \(\partial T_2^{-1}\) intersects \(\partial I_{23}^{-1} \cap \partial T_\lambda\) maps to \(C_3 = \frac{7}{12}\), and \(C_2\) maps to \(\frac{2}{3}\). Since the second free corner of \(T_3^{-1}\) lies between \(\frac{2}{3}\) and \(\frac{5}{6}\), we conclude that the second free corner of \(T_{23}^{-2}\) lies between the point where \(\partial T_2^{-1}\) intersects \(\partial I_{23}^{-1} \cap \partial B_\lambda\) and the point \(C_2\). We now have a topological invariant, which we call a
Figure 6. A model of the dynamical plane for $\lambda$ taken from the $\frac{1}{3}$-Mandelbrot set when $n = d = 2$.

chain of preimages of $T_{\lambda}$. When the critical point $c_i$ has period $m$ and itinerary $S(c_i) = is_1...s_{m-1}$, we consider a chain of preimages of $T_{\lambda}$ to be the collection $\{T_{\lambda}, T_{\lambda}^{-1}, T_{\lambda}^{-2}, ..., T_{\lambda}^{-m}\}$.

The structure of $T_{23}^{-2}$ and the chain of preimages of $T_{\lambda}$ are illustrated in Figure 7. We observe that the structure of this model corresponds to the structure exhibited by the Julia set in Figure 4.

Consider the map $F_{\mu}$ associated to the parameter $\mu$ taken from the main cardioid of a different accessible Mandelbrot set. If there exists a homeomorphism between $J(F_{\lambda})$ and $J(F_{\mu})$ which sends $\partial B_{\lambda}$ to $\partial B_{\mu}$ and $\partial T_{\lambda}$ to $\partial T_{\mu}$, then the homeomorphism must preserve the structure of this chain of preimages of $T_{\lambda}$. It is clear that for any map with critical orbit of period $m$, the chain of preimages contains $T_{\lambda}$ and $m$ of its preimages. Therefore, since $\frac{1}{3}$ is the only period 2 angle in the upper halfplane, we immediately conclude that there is no such homeomorphism between the Julia set associated to a map from the $\frac{1}{3}$-Mandelbrot set and the Julia set of a map taken from any other main cardioid in the upper halfplane. We recall, however, that maps taken from the $\frac{1}{3}$-Mandelbrot set and maps taken from the $\frac{2}{3}$-Mandelbrot set do have conjugate dynamics on their Julia sets.

Next, we examine the three accessible Mandelbrot sets in the upper halfplane that have critical orbits of period 3. These are the $\frac{1}{7}$, $\frac{2}{7}$, and $\frac{3}{7}$-Mandelbrot sets. We will see that the structure of the chain of preimages
Figure 7. The chain of preimages of $T_\lambda$ for $\lambda$ taken from the $\frac{1}{3}$-Mandelbrot set when $n = d = 2$. The highlighted portion of $\partial I^{-1}_{23}$ maps to the highlighted portion of $\partial I_3$ under $F_\lambda$.

of $T_\lambda$ is uniquely determined by the itinerary of the the periodic critical point which is, in turn, unique to the chosen Mandelbrot set.

Take, for example, $\lambda$ at the center of the main cardioid of the $\frac{2}{7}$-Mandelbrot set. We know the angles associated to the critical orbit are $\frac{2}{7} \mapsto \frac{4}{7} \mapsto \frac{1}{7} \mapsto \frac{2}{7}$. Since $\frac{2}{7}$ corresponds to the second iterate of the periodic critical point associated to the cycle, we know that the Fatou component containing the periodic critical point must lie at the landing point of the $\frac{1}{7}$-dynamical ray. This implies that $C_3$ contains the periodic critical point, so $h_\lambda(C_3) = \frac{1}{7}$. By symmetry, we compute that $h_\lambda(C_1) = \frac{1}{14}$, $h_\lambda(C_2) = \frac{9}{28}$, and $h_\lambda(C_0) = \frac{23}{28}$. This again divides the unit circle into four intervals.

\[
h_\lambda(\partial B_\lambda \cap I_1) = \begin{pmatrix} 1 \\ 14 \end{pmatrix}, \begin{pmatrix} 9 \\ 28 \end{pmatrix}
\]

\[
h_\lambda(\partial B_\lambda \cap I_2) = \begin{pmatrix} 9 \\ 28 \end{pmatrix}, \begin{pmatrix} 4 \\ 7 \end{pmatrix}
\]

\[
h_\lambda(\partial B_\lambda \cap I_3) = \begin{pmatrix} 4 \\ 7 \end{pmatrix}, \begin{pmatrix} 23 \\ 28 \end{pmatrix}
\]

\[
h_\lambda(\partial B_\lambda \cap I_0) = \begin{pmatrix} 23 \\ 28 \end{pmatrix}, \begin{pmatrix} 1 \\ 14 \end{pmatrix}.
\]
We now assign an itinerary to \( C_3 \) according to the behavior of \( h_\lambda(C_3) \) with respect to these four intervals. Recall that the forward orbit of the periodic dynamical ray landing at \( C_3 \) under doubling is given by \( \frac{1}{3} \mapsto \frac{2}{3} \mapsto \frac{1}{3} \mapsto \frac{2}{3} \). This tells us that \( C_3 \in \partial I_3 \), \( F_\lambda(C_3) \in \partial I_1 \), \( F_\lambda^2(C_3) \in \partial I_1 \), and \( F_\lambda^3(C_3) = C_3 \in \partial I_3 \). We define the itinerary of \( C_3 \) as \( 33I \). This itinerary will be useful in constructing the chain of preimages of \( T_\lambda \) in the model of \( J(F_\lambda) \).

The chain will include \( T_\lambda, T_\lambda^{-1}, T_{31}^{-1} \), and \( T_{311}^{-3} \). To give a full description of the structure of the chain, we determine the location of the corners of each element of the chain. We already know that the four corners of \( T_\lambda \) lie on \( \partial B_\lambda \) and that \( T_{31}^{-1} \) has two corners on \( \partial B_\lambda \) and two corners on \( \partial T_\lambda \).

\( T_{31}^{-1} \) divides \( I_3 \) into four subsectors. \( T_{31}^{-2} \) is contained in \( I_{31}^{-1} \) and the four corners of \( T_{31}^{-2} \) lie on \( \partial I_{31}^{-1} \). Exactly two corners of \( T_{31}^{-2} \) lie in \( \partial T_{31}^{-1} \), so the two free corners of \( T_{31}^{-2} \) must lie in \( \partial I_{31}^{-1} \cap (\partial B_\lambda \cup \partial T_\lambda) \). As in our previous example, the order of points along this curve is preserved under \( F_\lambda \).

The point \( \partial I_{31}^{-1} \cap \partial T_\lambda \cap \partial T_{31}^{-1} \) maps to \( C_1 \). The free corners of \( T_{31}^{-2} \) map to the free corners of \( T_{11}^{-1} \) in \( \partial B_\lambda \). \( C_3 \) maps to a point on the boundary of \( I_{11}^{-1} \). \( F_\lambda(C_3) \) lies between \( C_1 \) and the free corners of \( T_{11}^{-1} \) along the curve \( \partial B_\lambda \cap \partial I_1 \). Therefore, \( C_3 \) must lie between the point \( \partial I_{31}^{-1} \cap \partial T_\lambda \cap \partial T_{31}^{-1} \) and the free corners of \( T_{31}^{-2} \) along the curve \( \partial I_{31}^{-1} \cap (\partial B_\lambda \cup \partial T_\lambda) \), see Figure 8.

\( T_{311}^{-3} \) has two corners on \( \partial T_{31}^{-2} \) and two free corners which map onto the two free corners of \( T_{11}^{-2} \). Since \( C_3 \) has period 3, one free corner of \( T_{311}^{-3} \) must be \( C_3 \) (in other words, \( C_3 \) is the preimage of one of the free corners of \( T_{11}^{-2} \)). It remains to find the location of the second free corner of \( T_{311}^{-3} \), which must lie in \( \partial I_{311}^{-2} \cap (\partial B_\lambda \cup \partial T_\lambda) \).

By symmetry, \( T_{11}^{-2} \) has the same structure as \( T_{31}^{-2} \). This allows us to conclude that \( T_{11}^{-2} \) also has its two free corners on \( \partial B_\lambda \). The free corners of \( T_{11}^{-2} \) then map onto the free corners of \( T_{11}^{-1} \) and the free corners of \( T_{11}^{-1} \) map onto \( C_2 \) and \( C_3 \).

In particular, we note that we must move clockwise along \( \partial B_\lambda \cap \partial I_1 \) to get from \( F^{-1}(C_2) \) to \( F^{-1}(C_3) \) and clockwise along \( \partial B_\lambda \cap \partial I_1^{-1} \) to get from \( F^{-2}(C_2) \) to \( F^{-2}(C_3) \). Similarly, we move clockwise along \( \partial I_{311}^{-2} \cap (\partial B_\lambda \cup \partial T_\lambda) \) to get from \( F^{-3}(C_3) = C_3 \) to the second free corner of \( T_{311}^{-3} \), see Figure 8. The complete chain of preimages of \( T_\lambda \) for this example is illustrated in Figure 9.

We then ask whether the chains of preimages of \( T_\lambda \) for maps taken from other accessible Mandelbrot sets with period 3 critical orbits will necessarily have the same structure. A quick glance at the numerically generated
Figure 8. Locating the corners of $T_{31}^{-2}$ and $T_{31}^{-3}$ for $\lambda$ taken from the $\frac{4}{3}$-Mandelbrot set when $n = d = 2$. In the top figure, the highlighted portion of $\partial I_{31}^{-1}$ maps onto the highlighted portion of $\partial I_1$. In the bottom figure, the highlighted portion of $\partial I_{31}^{-2}$ maps onto the highlighted portion of $\partial I_{31}^{-1}$. 
Figure 9. The final chain of preimages of $T_\lambda$ for $\lambda$ taken from the $\frac{2}{7}$-Mandelbrot set when $n = d = 2$.

dynamical planes for maps taken from the $\frac{1}{7}$, $\frac{2}{7}$, and $\frac{3}{7}$-Mandelbrot sets (see Figure 10) might suggest that a homeomorphism exists between their Julia sets. However, using the same method as before, we construct the chains of preimages of $T_\lambda$ for the $\frac{1}{7}$ and $\frac{3}{7}$-Mandelbrot sets. It is clear from the structures of these chains (also shown in Figure 10) that no two of these maps can have homeomorphic Julia sets. We conclude that maps taken from the main cardioids of distinct accessible period-3 Mandelbrot sets cannot have conjugate dynamics on their Julia sets, unless these cardioids are complex conjugates of one another.

Next, we consider maps with critical orbits of period greater than 3. Our method for constructing chains will continue to prove useful. However, as the period increases, we see that constructing the chains becomes more complicated. In our previous examples, the free corners of each $T^{-k}$ have fallen either on $\partial B_\lambda$ or $\partial T_\lambda$, but this is not always the case. Sometimes, the free corners of a $T^{-k}$ will fall on the boundary of some $T^{-j}$ with $j < k$.

Take for example, $\lambda$ at the center of the main cardioid of the $\frac{6}{15}$-Mandelbrot set. We know the angles associated to the critical orbit are $\frac{3}{5} \mapsto \frac{1}{5} \mapsto \frac{2}{5} \mapsto \frac{4}{5} \mapsto \frac{3}{5}$. Since $\frac{3}{5}$ corresponds to the second iterate of the periodic critical point associated to the cycle, we know that the Fatou component containing the periodic critical point must lie at the landing point of the $\frac{2}{5}$ dynamical ray. This implies that $C_3$ contains the periodic critical point, so $h_\lambda(C_3) = \frac{2}{5}$. By symmetry, we compute that $h_\lambda(C_1) = \frac{1}{15}$, $h_\lambda(C_2) = \frac{7}{25}$, and $h_\lambda(C_0) = \frac{17}{25}$. This again divides the unit.
Figure 10. The dynamical planes and the corresponding Julia set models for maps taken from the $\frac{1}{7}$, $\frac{2}{7}$, and $\frac{3}{7}$-Mandelbrot sets, respectively, when $n = d = 2$. At first glance, these Julia sets may appear homeomorphic. Comparison of their models clearly shows that this is not the case.
circle into four intervals.

\[
\begin{align*}
    h_\lambda (\partial B_\lambda \cap I_1) &= \begin{bmatrix} 1/10 & 7/20 \end{bmatrix} \\
    h_\lambda (\partial B_\lambda \cap I_2) &= \begin{bmatrix} 7/20 & 3/5 \end{bmatrix} \\
    h_\lambda (\partial B_\lambda \cap I_3) &= \begin{bmatrix} 3/5 & 17/20 \end{bmatrix} \\
    h_\lambda (\partial B_\lambda \cap I_0) &= \begin{bmatrix} 17/20 & 1/10 \end{bmatrix}.
\end{align*}
\]

Since \( \frac{3}{5} \) has period 4 under doubling, we conclude that \( C_3 \) contains the periodic critical point and its itinerary is given by \( S(C_3) = 3123 \).

\( C_3 \) must lie within \( I_{31}^{-1} \), which implies that \( C_3 \) falls between the free corners of \( T_{31}^{-2} \). Thus one free corner of \( T_{31}^{-2} \) lies on \( \partial B_\lambda \) and the other lies on \( \partial T_\lambda \). \( C_3 \) must also lie within \( I_{312}^{-2} \), which implies that both free corners of \( T_{31}^{-3} \) lie on \( \partial B_\lambda \) between \( C_3 \) and the free corner of \( T_{31}^{-2} \) that lies on \( \partial B_\lambda \). So far, our method has progressed just as in our previous examples. Next, we know that \( C_3 \) is one corner of \( T_{312}^{-4} \) and we need only find the location of a single free corner of \( T_{31}^{-4} \). This free corner lies on \( \partial I_{3123}^{-3} \). Recall that each preimage of \( T_\lambda \) has exactly two corners on a previous preimage of \( T_\lambda \), this implies that the free corner of \( T_{312}^{-4} \) cannot lie in the section of \( \partial I_{3123}^{-3} \) that intersects \( \partial T_{312}^{-3} \). Thus, the free corner lies either on \( \partial B_\lambda \cap \partial I_{3123}^{-3} \) or \( (\partial T_\lambda \cup \partial T_{31}^{-2}) \cap \partial I_{3123}^{-3} \). \( F_\lambda(C_3) \) is the corner of \( T_3^{-1} \) in \( \partial T_\lambda \) closest to \( C_0 \). This implies that the free corner of \( T_{3123}^{-4} \) lies in \( (\partial T_\lambda \cup \partial T_{31}^{-2}) \cap \partial I_{3123}^{-3} \). To complete the model, we just determine whether this free corner lies on \( \partial T_\lambda \) or \( \partial T_{31}^{-2} \). The point \( \partial T_\lambda \cap \partial T_{31}^{-2} \) divides these two curve segments. By comparing the forward images of this point to the forward images of the free corner of \( T_{3123}^{-4} \) and using the fourfold symmetry of the Julia set, we can determine the that the free corner lies on \( \partial T_{31}^{-2} \). The model of this Julia set can be seen in Figure 11.

7. Conclusion

There is a rational parameter ray corresponding to each accessible Mandelbrot set. This rational angle corresponds to a unique itinerary for the periodic critical point of a map taken from center of the main cardioid of this Mandelbrot set. The itinerary determines the model for the Julia set of the map. Thus, each ray defines a unique model. Likewise, given a model, one can determine the unique corresponding itinerary. This allows us to conclude the following:

For \( n = d = 2 \), given parameters \( \lambda \) and \( \mu \) from the main cardioids of
Figure 11. A model of the dynamical plane for $\lambda$ taken from the $\frac{6}{15}$ Mandelbrot set when $n = d = 2$.

different accessible Mandelbrot sets, there exists a homeomorphism taking $\partial B_\lambda$ to $\partial B_\mu$ and $\partial T_\lambda$ to $\partial T_\mu$ if and only if $\mu = \overline{\lambda}$.

In ongoing work, we consider cases of $n = d > 2$ and we conjecture that the same result will hold. We have found that, when $n$ is odd, the construction of the models can be more challenging because there are always two periodic critical points. Sometimes these points lie on the same orbit of period $2n$ and other times each periodic critical point has a unique orbit of period $n$. Construction of these models is made easier by considering a map which is semi-conjugate to $F_\lambda$ but has a single critical point.

We reiterate the distinction between our results those for the principal Mandelbrot sets. Each map taken from the main cardioid of a principal Mandelbrot set is known to have a Checkerboard Julia set. The Julia sets corresponding to maps taken from the main cardioids of distinct principal Mandelbrot sets are known to be homeomorphic and, in some cases, these maps exhibit conjugate dynamics on their Julia sets. In contrast, when we consider maps taken from main cardioids of distinct accessible Mandelbrot sets of period $m \geq 2$, we have seen that the corresponding Julia sets cannot be homeomorphic and thus, such maps can never exhibit conjugate dynamics on their Julia sets.
References


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